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Hamilton–Jacobi Quantization of Systems With Time-Dependent Constraints

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Hamilton–Jacobi formalism is used to investigate time-dependent constraint systems. It is proved that the generalization of Dirac's canonical quantization method in the non-stationary case can be obtained naturally in Hamilton–Jacobi formalism. The example of the relativistic particle in a plane wave is analyzed in detail.

KEY WORDS: Hamilton approach; constraint system; quantization.

1. INTRODUCTION

The direct way to quantize a system with constraints is to use canonical quantization based on the Hamiltonian formalism of the classical theory (Dirac, 1964; Hanson *et al.*, 1976; Henneaux, 1985; Henneaux and Teiteilbom, 1992; Sundermeyer, 1982).

A new quantization method for constrained systems was initiated by one of us (Güler, 1987, 1989, 1992) and the method was generalized to singular systems with higher order Lagrangians as well as to systems containing elements of the Berezin algebra (Pimentel and Teixeira, 1998; Pimentel *et al.*, 1996, 1998).

The advantage of the Hamilton–Jacobi formalism is that we have no difference between first and second class constraints and no gauge fixing term is necessary because the gauge variables are separated in the process of constructing an integrable system of total differential equations. In addition the action provided by the formalism can be used in the process of the path integral quantization method (Baleanu and Güler, 2001a,b,c). In this formalism the phase space is enlarged from

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the beginning (Güler, 1987, 1989, 1992) and the integrability conditions are the same as Dirac's consistency conditions (Pimentel *et al.*, 1998).

Recently the time-dependent Schrödinger equation for systems invariant under the reparametrization of time was debated in Tkach *et al.* (1999); its BRST treatment was obtained (Marnelius and Sandstrom, 2000) and the reparametrization invariance was treated as a gauge symmetry (Fülöp *et al.*, 1999).

The quantization of a relativistic particle in a plane wave was analyzed in Gavrilov and Gitman (1993) as well as the gauge invariance for generally covariant systems was investigated in Henneaux *et al.* (1992). The connection between the canonical and the path integral formulations of the quantum mechanics of a relativistic particle was developed (Hartle and Kuchar, 1986).

On the other hand an intriguing problem is to apply the Hamilton–Jacobi formalism when the Hamiltonians are not linearly independent and to compare the obtained results with those provided by other methods.

The main aim of this paper is to investigate the quantization of systems with time-dependent constraints by using Hamilton–Jacobi formulation.

The organization of the paper is as follows: In Section 2 the Hamilton–Jacobi formulation and its connection with Dirac's modification method of time-dependent second class constraints are presented. In Section 3 the reparametrization invariance theories are analyzed by using Hamilton–Jacobi formalism. Our conclusions are presented in Section 4.

2. HAMILTON–JACOBI FORMALISM

In this formulation we start with a singular lagrangian of Hessian matrix of rank n-r. The formalism leads us to the following Hamiltonians (Güler, 1992):

$$H'_\alpha = H_\alpha(t_\beta, q_a, p_a) + p_\alpha, \tag{1}$$

where $\alpha, \beta = n - r + 1, \ldots, n$ and $a = 1, \ldots, n - r$. The usual Hamiltonian H_0 is defined as

$$\begin{aligned} H_0 = {} & -L(t, q_i, \dot{q}_\nu, \dot{q}_a = w_a) + p_a w_a \\ & + \dot{q}_\mu p_\mu \,|_{p_\nu = -H_\nu}, \quad \nu = 0, n - r + 1, \ldots, n, \end{aligned} \tag{2}$$

which is independent of $\dot{q}_\mu$. Here $\dot{q}_a = \frac{dq_a}{d\tau}$, where τ is a parameter. The equations of motion are obtained as total differential equations in many variables as follows:

$$\begin{aligned} dq_a &= \frac{\partial H'_\alpha}{\partial p_a}\, dt_\alpha, \qquad dp_a = -\frac{\partial H'_\alpha}{\partial q_a}\, dt_\alpha, \\ dp_\mu &= -\frac{\partial H'_\alpha}{\partial t_\mu}\, dt_\alpha, \quad \mu = 1, \ldots, r, \end{aligned} \tag{3}$$

$$dz = \left(-H_\alpha + p_a \frac{\partial H'_\alpha}{\partial p_a}\right) dt_\alpha, \tag{4}$$

where $z = S(t_\alpha, q_a)$.

One should notice that although we have started with n generalized coordinates q_i and generalized velocities $\dot{q}_i$ to pass to canonical formulation we have to treat some generalized momenta-dependent and corresponding generalized coordinates as free parameters. Thus, we have a phase space of lower dimension. But this is not sufficient because the equations of motion are total differential equations and we have integrability conditions. In other words Eqs. (3) and (4) are integrable iff $dH'_\alpha = 0$. Some of these conditions could be satisfied identically and the rest may cause new constraints. Again using the same test, the additional constraints might arise. As a result, it may happen that we have a set of constraints, which are in involution, and an integrable system. Every new constraint leads us to reduce the dimension of the phase space. Thus, we may have constraints other than (1) and (2). The essence of the theory is to express constraints in the form (1) and (2).

2.1. Integrability Conditions

Let us define the linear operators X_α as (Güler, 1992)

$$X_\alpha f = [f, H'_\alpha] = \frac{\partial f}{\partial q_\beta}\frac{\partial H'_\alpha}{\partial p^\beta} - \frac{\partial f}{\partial p^\beta}\frac{\partial H'_\alpha}{\partial q_\beta} + \frac{\partial f}{\partial \chi_\alpha}. \tag{5}$$

Lemma. *A system of differential equations* (3) *is integrable iff*

$$[H'_\alpha, H'_\beta] = 0. \tag{6}$$

Proof: If we suppose that (6) is satisfied, then we have

$$(X_\alpha, X_\beta) f = (X_\alpha X_\beta f - X_\beta X_\alpha) f = X_\alpha[f, H'_\beta] - X_\beta[f, H'_\alpha]. \tag{7}$$

Making use of Jacobi's identity we obtain

$$(X_\alpha, X_\beta) f = [f, [H'_\beta, H'_\alpha]]. \tag{8}$$

Using (6) and (8) we find

$$(X_\alpha, X_\beta) = 0. \tag{9}$$

Conversely, if the system is complete, then (8) is fulfilled for any α and β and we obtain

$$[H'_\alpha, H'_\beta] = 0. \tag{10}$$

□

In the concrete applications it may happen that H'_α are not in involution, for example, when the second class constraints appear (Dominichi *et al.*, 1984; Rabei and Güler, 1992). In this case the associated total differential system of equations is not integrable and the modification of the phase space is required (Baleanu and Güler, 2000; Rabei and Güler, 1992). If we try to apply the canonical quantization method we find easily the Dirac's brackets action on the extended phase space (Baleanu and Güler 2000).

2.2. Modified Dirac's Method of Time-Dependent Second Class Constraints Theories

In the following we assume that we have a constrained system having $\Phi(\eta, t) = 0$ second class constraints, where $\eta = (q, p)$, and it can explicitly depend on time t. In order to keep the equations of motion in the usual form in Gavrilov and Gitman (1993) a momenta ϵ conjugate to t was introduced. In this manner the Poisson brackets are considered in the extended phase space $(q, p, t, \epsilon) = (\eta, t, \epsilon)$ as follows:

$$\dot{\eta} = \{\eta, H + \epsilon\}_{D(\Phi)}, \qquad \Phi(\eta, t) = 0, \tag{11}$$

where H represents the Hamiltonian of the system and the braces denote the Dirac's bracket corresponding to the set of contraints Φ (for more details see Gavrilov and Gitman, 1993, and the references therein).

In the Hamilton–Jacobi formulation the Hamiltonian $H'_0 = p_0 + H_c$ is in the form similar to $H + \epsilon$ Thus, we conclude that the generalization of Dirac's quantization (Gavrilov and Gitman, 1993; Gitman and Tyutin, 1990) method emerges naturally in Hamilton–Jacobi formalism.

3. THE REPARAMETRIZATION INVARIANCE

The reparametrization invariant action corresponding to a spinless particle in an electromagnetic field of a plane wave oriented along the x axis is (Gavrilov and Gitman, 1993)

$$S = -\int m\sqrt{\dot{x}^2} + e\dot{x}A\,d\tau = -\int \left[m\sqrt{2\dot{x}_-\dot{x}_+ - (\dot{x}_\perp)^2} + e\dot{x}^a A_a(x_-)\right] d\tau \tag{12}$$

Here $A_\mu = (0, A^a, 0)$, $x_\pm = (x^0 \pm x^3)/\sqrt{2}$, $x_\perp = (x^a)$, $a = 1, 2$.

$$\pi_\pm = \frac{\partial L}{\partial x_\pm} = -\frac{m\dot{x}_\pm}{\sqrt{2\dot{x}_-\dot{x}_+ - (\dot{x}_\perp)^2}} \tag{13}$$

$$\pi_a = \frac{\partial L}{\partial \dot{x}^a} = -\frac{m\dot{x}^a}{\sqrt{2\dot{x}_-\dot{x}_+ - (\dot{x}_\perp)^2}} - eA_a(x_-) \tag{14}$$

We mention that (13) are not suitable for application of Hamilton–Jacobi formalism simply because the velocities appear. Since the rank of the Hessian matrix is 3 we conclude that not all the momenta are independent. On the other hand we can easily verify that sgn $\dot{x}_- = -\text{sgn}\,\dot{\pi}_+ = \zeta$ and we have a primary constraint (Gavrilov and Gitman, 1993).

$$H_1' = \pi_- - \frac{[\pi_a + eA_a(x_-)]^2 + m^2}{2\pi_+} = 0, \quad \pi_\pm \neq 0. \tag{15}$$

The canonical Hamiltonian becomes

$$H_c = 2\lambda\zeta H_1', \quad \lambda = |\dot{x}_-|. \tag{16}$$

In Hamilton–Jacobi formalism we have at this stage two Hamiltonians

$$H_0' = p_0 + H_c, \qquad H_1' = \pi_- - \frac{[\pi_a + eA_a(x_-)]^2 + m^2}{2\pi_+} = 0, \quad \pi_\pm \neq 0, \tag{17}$$

But they are not linearly independent. In other words the theory is degenerate. On the other hand x_- and τ look like gauge variables.

Let us now check the integrability conditions. If the variations of H_1' are zero, then automatically the variations of H_0' are zero.

From $dH_1' = 0$ we obtain

$$d\pi_- = -e\dot{x}^a A_a' \, d\tau \tag{18}$$

and it is one of the Euler–Lagrange equations.

Now the system of total differential equations is integrable and the remaining total differential equations are

$$dx_- = 2\lambda\zeta \, d\tau, \qquad d\pi_+ = 0, \qquad d\pi_a = 0. \tag{19}$$

Integrating $dx_- = \lambda\zeta \, d\tau$, we get $x_- - 2\lambda\zeta\tau = c$, where c is a real constant.

Taking into account that $d\lambda = 0$ and $\lambda = \frac{1}{2}$, we reobtain the same gauge condition from Gavrilov and Gitman (1993). In other words x_- and τ are not linearly independent. Then the corresponding *Schrödinger* equation becomes

$$i\frac{\partial\Psi}{\partial x_-} = -\frac{\left[-\frac{i\partial}{\partial x^a} - eA^a(x_-)\right]^2 + m^2}{2\pi_+}\Psi. \tag{20}$$

4. CONCLUSIONS

Despite many attempts to developed the Hamilton–Jacobi formalism it contains unexplored parts, e.g., the case when the constraints are linearly dependent, they are not in the form (1) or they are second class. The main problem is whether the results provided by Hamilton–Jacobi and Dirac's are in agreement with each other. Since the phase space is extended from the beginning in Hamilton–Jacobi formalism and in addition the integrability conditions are the same as Dirac's consistency conditions, we conclude that both formalism leads us to the same canonical quantization procedure. Hamilton–Jacobi formalism can be applied without difficulty when the constraints are dependent on time; the main problem is to keep its physical significance, more exactly all the constraints must be in the form (1).

The reparametrization invariance theory is problematic for Hamilton–Jacobi formalism because we have two Hamiltonians and they are not independent. Using the integrability conditions we reobtained the gauge condition imposed in Gavrilov and Gitman (1993) as one of the total differential equations. Once we found that p_0 and x_- are related the Schrödinger's equation was obtained.

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